Algebra Qualifying Exam January 2015

The exam consists of ten problems; each problem is worth 10 points.

1. Decide if the statement below is true or false. Prove or give a counterexample.

"If H_1, H_2 are groups and $G = H_1 \times H_2$, then every subgroup of G is of the form $K_1 \times K_2$, with K_i a subgroup of H_i for $i = 1, 2$."

2. Let G be a group and let H be a subgroup of G. Prove that the following two statements are equivalent:

(i) $x^{-1}y^{-1}xy \in H$ for all $x, y \in G$

(ii) H is normal in G and G/H is abelian.

3. Let $p_1 < p_2 < p_3$ be distinct prime numbers, and let G be a group of order $p_1p_2p_3$. Prove that G is not a simple group.

4. Let R be a ring and let M, N, K be R-modules. Let $f : M \to N$ be an R-module homomorphism.

(a) Prove that $f_* : \mathrm{Hom}_R(K, M) \to \mathrm{Hom}_R(K, N)$ is an R-module homomorphism, where $f_*(\alpha) = f \circ \alpha$ for all $\alpha \in \mathrm{Hom}_R(K, M)$ (recall that the operations on $\mathrm{Hom}_R(K, M)$ are defined by $(\alpha+\beta)(x) = \alpha(x)+\beta(x), (r \cdot \alpha)(x) = r\alpha(x)$ for $\alpha, \beta \in \mathrm{Hom}_R(K, M), x \in K, r \in R$)

(b) Prove that if f is 1-1 then f_* is also 1-1.

(c) Assume that R is a domain, I is a proper non-zero ideal of R, $M = R, N = K = R/I$, and $f : R \to R/I$ is the canonical projection that takes each element to its congruence class. Prove that f_* is not onto (even though f is onto).

5. Let $u = \sqrt[4]{2}$, and let D_4 be the dihedral group of rigid motions of a square. Recall that D_4 can be described by generators and relations as follows: $D_4 =< x, y \,|\, x^4 = e, y^2 = e, yx = x^3y >$ where x, y denote the generators of the group, and e denotes the identity element of the group.

a. Prove that $\mathrm{Gal}(\mathbf{Q}(u, i)/\mathbf{Q})$ is isomorphic to D_4.

b. Using the isomorphism from (a), what is the subgroup of D_4 corresponding to $Q(u)$ under Galois correspondence?

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6. Let F be a field of characteristic p, where p is a prime number, and let $c \in F$, $f(X) = X^p - X - c \in F[X]$. Show that $f(X)$ factors completely into linear factors in $F[X]$, or else $f(X)$ is irreducible in $F[X]$ (hint: show that if a is a root of $f(X)$ in a splitting field, then $a + 1$ is also a root of $f(X)$)

7.

(a) Let R be a PID (Principal Ideal Domain). Prove that every non-zero prime ideal of R is maximal.

(b) Give an example of a commutative ring R and a non-zero prime ideal I that is not maximal.

(c) Let K a field which is NOT algebraically closed. Give an example of a maximal ideal of the ring $R = K[X, Y]$ which is NOT of the form $(X - a, Y - b)$ with $a, b \in K$ (recall that a field L is algebraically closed if every non-constant polynomial $f(X) \in L[X]$ has at least a root in L; this problem is asking you to prove the converse of the Hilbert Nullstellesatz Theorem, which is easier than the actual theorem).

8. Prove that every finite group is isomorphic to a subgroup of some symmetric group S_n (for some positive integer n).

9. Let R be a commutative ring. Recall that an R-module M is called a *free* R-module if and only if it is isomorphic to R^n for some positive integer n. Let I be a non-zero proper ideal of R. Recall that we can view I as an R-module using the multiplication in R as scalar multiplication.

Prove that I is a free R-module if and only if I is a principal ideal and $\mathrm{Ann}_R(I) = (0)$ ($\mathrm{Ann}_R(I)$ means the set $\{x \in R \mid xa = 0 \, \forall a \in I\}$).

10. (a) Find the minimal polynomial of $\sqrt{4 + \sqrt{7}}$ over $\mathbb{Q}$.

(b) Find the Galois group of that polynomial's splitting field over $\mathbb{Q}$ (hint: show that $\sqrt{4 + \sqrt{7}} = (\sqrt{2} + \sqrt{14})/2$).