Qualifying exam in analysis, January 2019

Instructions: Write your identifying number legibly on each sheet of paper. Write only on one side of each sheet of paper. Each of the Questions 1-8 are worth 10 points and Question 9 is worth 20 points. Even though the questions are broken into parts and partial credit will be given, the number of questions that will be answered completely will also be taken into account by the examiners. The Lebesgue measure will be denoted by λ. If A is a subset of $\mathbb{R}$ and the measure is not specified, then for $1 \leq p \leq \infty$, the spaces $\mathcal{L}^p(A)$ and $L^p(A)$ as well as the norms $\|\cdot\|_p$ are considered with respect to the Lebesgue measure. You can quote without proof any of the standard theorems covered in Math 703-704, but do indicate why the relevant hypotheses hold. You are also allowed to rely on parts of this exam that you have not solved in order to solve other parts of this exam.

1)a) (5 pts) For every $n \in \mathbb{N}$ let $g_n : [0,1] \to [0,\infty)$ be a continuous function such that the sequence $(g_n)_{n\in\mathbb{N}}$ is pointwise monotonically decreasing (i.e. $g_n(x) \geq g_{n+1}(x)$ for every $n \in \mathbb{N}$ and every $x \in [0,1]$) and converges pointwise to 0. For $\epsilon > 0$ and $n \in \mathbb{N}$ let $E(n,\epsilon) = \{x \in [0,1] : g_n(x) < \epsilon\}$. Prove that for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $E(n,\epsilon) = [0,1]$ for all $n \geq n_0$.
b) (2.5 pts) State what it means that a sequence of functions $(f_n)_{n\in\mathbb{N}}$ on $[0,1]$ converges uniformly to a function f.
c) (2.5 pts) Prove that if $(f_n)_{n\in\mathbb{N}}$ is a pointwise monotonically increasing sequence of continuous real valued functions on $[0,1]$ which converges pointwise to a continuous function f then it converges uniformly.

2) A function $\phi : [0,1] \to \mathbb{R}$ is called convex if $\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$ for all $x, y \in [0,1]$ and for all $t \in [0,1]$.
a) (3 pts) Prove that no convex function $\phi : [0,1] \to \mathbb{R}$ has the following property: There exists a point $x_0 \in [0,1)$, a strictly decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in $(0,1]$ with $x_n \to x_0$, and an $\epsilon > 0$ such that $\phi(x_n) \geq \phi(x_0) + \epsilon$ for every $n \in \mathbb{N}$.
b) (5 pts) Prove that no convex function $\phi : [0,1] \to \mathbb{R}$ has the following property: There exists a point $x_0 \in (0,1)$, a strictly decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in $(0,1]$ with $x_n \to x_0$, and an $\epsilon > 0$ such that $\phi(x_n) \leq \phi(x_0) - \epsilon$ for every $n \in \mathbb{N}$.
c) (2 pts) Prove that every convex function $\phi : [0,1] \to \mathbb{R}$ is continuous on $(0,1)$.

3) Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) = 1$. Let $f : \Omega \to [0,1]$ be an Σ-measurable function and $\phi : [0,1] \to \mathbb{R}$ be a convex function, (see the statement of Question 2) for the definition of a convex function).
a) (2 pts) Prove that for every $x_0 \in [0,1]$ and for every $\epsilon, \delta > 0$ such that $0 \leq x_0 - \delta <$

$x_0 < x_0 + \epsilon \leq 1$, we have $\frac{\phi(x_0)-\phi(x_0-\delta)}{\delta} \leq \frac{\phi(x_0+\epsilon)-\phi(x_0)}{\epsilon}$.
b) (2 pts) Prove that for every $x_0 \in (0,1)$ there exists $m \in \mathbb{R}$ such that $m(x-x_0)+\phi(x_0) \leq \phi(x)$ for all $x \in [0,1]$. Is the same claim valid for $x_0 = 0$ or $x_0 = 1$?
c) (2 pts) Prove that the composition $\phi \circ f$ is Σ-measurable.
d) (2 pts) Prove that $\phi(\int_\Omega f d\mu) \leq \int_\Omega \phi \circ f d\mu$.
e) (2 pts) Prove that $\int_\Omega f d\mu \ln(\int_\Omega f d\mu) \leq \int_\Omega f \ln(f) d\mu$ where we define $0 \ln 0 = 0$.

4)a) (2 pts) Let $g \in \mathcal{L}^2([0,1],\lambda)$ and $\epsilon > 0$. Prove that $\lambda\{x \in [0,1] : |g(x)| > \epsilon\} \leq \frac{1}{\epsilon^2}\int_0^1 g^2 d\lambda$.
b) (3 pts) For $n \in \mathbb{N}$ let $f_n : [0,1] \to [-1,1]$ be defined by $f_n(x) = \operatorname{sign}(\sin(2^n \pi x))$, where $\operatorname{sign}(x) = x/|x|$ if $x \neq 0$ and $\operatorname{sign}(0) = 0$. Compute $\int_0^1 f_k(x) f_\ell(x) d\lambda(x)$ when $k \neq \ell$ and when $k = \ell$.
c) (5 pts) For $n \in \mathbb{N}$ let $g_n : [0,1] \to [-1,1]$ be defined by $g_n = \frac{1}{n}\sum_{k=1}^n f_k$ (where f_k is defined in part b)). For $\epsilon > 0$ show that $\lambda\{x \in [0,1] : |g_n(x)| > \epsilon\} \to 0$ as $n \to \infty$.

5)a) (5 pts) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real valued differentiable functions on $(0,1)$ such that the sequence $(f'_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on $(0,1)$. Let $\epsilon > 0$. Show that there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ and for all $x, x_0 \in (0,1)$ with $x \neq x_0$ we have that

$$\left| \frac{f_m(x) - f_m(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| < \epsilon.$$

Explain how the fact that (f'_n) is uniformly Cauchy, (instead of simply pointwise Cauchy), is used in your proof.
b) (5 pts) Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of real valued functions on $(0,1)$ and let g be a real valued function on $(0,1)$. State what it means that the sequence $(g_n)_{n\in\mathbb{N}}$ converges uniformly on $(0,1)$ to g; what it means that the sequence $(g_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on $(0,1)$; and prove that if $(g_n)_{n\in\mathbb{N}}$ converges uniformly on $(0,1)$ to g then $(g_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on $(0,1)$.

6)a) (2 pts) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real valued differentiable functions on $(0,1)$ such that $(f'_n)_{n\in\mathbb{N}}$ converges uniformly on $(0,1)$ to a function g, and $(f_n)_{n\in\mathbb{N}}$ converges pointwise on $(0,1)$ to a function f. Let $\epsilon > 0$. Show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x, x_0 \in (0,1)$ with $x \neq x_0$ we have that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| < \epsilon.$$

b) (2 pts) If f, g are two real valued functions on $(0,1)$, write down the ϵ-δ definition of the fact that "f is differentiable on $(0,1)$ and $f' = g$".

c) (6 pts) Assume the assumptions of Question 6)a). Prove that f is differentiable on $(0, 1)$ and $f' = g$.

7)a) (3 pts) If f is an entire function write the coefficients of the Taylor series of f as path integrals on the circle $C_R = \{z \in \mathbb{C} : |z| = R\}$ for some $R > 0$.
b) (7 pts) Prove that if f is an entire function which satisfies the inequality $|f(z)| \leq 2019|z|^{10}$ for all $z \in \mathbb{C}$ with $|z| \geq 17$ then f is a polynomial of degree at most 10.

8) Consider the function $f(z) = \frac{1}{\sin z} - \frac{1}{z}$.
a) (5 pts) Examine the type of singularity of f at 0.
b) (5 pts) Find all poles of f and compute the corresponding residues.

9) True or False. Prove, or give a counterexample.

a) (5 pts) If a sequence of real valued functions is everywhere differentiable and it converges uniformly then the sequence of their derivatives converges pointwise.

b) (5 pts) If a sequence of analytic functions on the closed unit disc and converges uniformly on the closed unit disc then the sequence of their derivatives converges uniformly on any compact subset of the open unit disc.

c) (5 pts) If a sequence of pointwise monotonically decreasing, continuous, real valued functions on $(0, 1)$ converges pointwise to 0 then it converges uniformly.

d) (5 pts) If the function f is continuous on a neighbourhood of 0 then $\lim_{r\to 0} \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z} dz = f(0)$ where C_r is the circle on $\mathbb{C}$ with center 0 and radius r.

e) (5 pts) If the function $f : [0, 1] \to \mathbb{R}$ is Lebesgue measurable and bounded, then $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.