Math708&709 – Foundations of Computational Mathematics

Qualifying Exam

August, 2013

Note: You must show all of your work to get a credit for a correct answer.

1. Given the following data for a function $f : \mathbb{R} \to \mathbb{R}$:

x	0	1	2
$f(x)$	-1	2	15

(a) Construct the quadratic interpolation polynomial $p_2(x)$ which interpolates the data.
(b) If the function being interpolated was in fact $f(x) = x^3 + 2x^2 - 1$, derive a tight upper bound on the error in using $p_2(x)$ as an approximation to $f(x)$ on $[0, 2]$.

2. This problem concerns orthogonal polynomials and Gaussian quadratures.
(a) Find $\{p_0, p_1, p_2\}$ such that p_i is a polynomial of degree i and these polynomials are orthogonal to each other on $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$.
(b) Find the points and weights $\{(x_i, w_i)\}_{i=1}^2$ of the 2-point Gaussian quadrature

$$\int_0^\infty f(x)e^{-x}dx \approx w_1 f(x_1) + w_2 f(x_2).$$

3. Consider the following Runge-Kutta method for solving the initial value problem $y' = f(t, y), y(0) = y_0$ where h is the time step size:

$$y_{n+1} = y_n + \alpha h f(t_n, y_n) + \frac{h}{2} f(t_n + \beta h, y_n + \beta h f(t_n, y_n)).$$

(a) For what values of $\{\alpha, \beta\}$ is the method consistent?
(b) For what values of $\{\alpha, \beta\}$ is the method stable?
(c) For what values of $\{\alpha, \beta\}$ is the method most accurate?

4. Consider the 3-step Adams-Bashforth method,

$$y_{n+1} = y_n + h\left[\frac{23}{12} f(t_n, y_n) - \frac{4}{3} f(t_{n-1}, y_{n-1}) + \frac{5}{12} f(t_{n-2}, y_{n-2})\right]$$

for solving the initial value problem $y' = f(t, y), y(0) = y_0$.
(a) Derive this method based on $y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t, y)\,dt$ and the polynomial interpolation approximation of f on t_n, t_{n-1}, t_{n-2}.
(b) Determine the order of accuracy of this linear multistep method.
(c) Is the method convergent? Justify your answer.

5. This problem concerns condition numbers and system stability.
(a) Let $\mathbf{A}$ be an $n \times n$ nonsingular matrix. We consider the solution of the linear

system $\mathbf{Ax} = \mathbf{b}$. Suppose we have an approximate solution $\mathbf{x}^*$ to the exact solution $\mathbf{x}$ of this system, and let $\mathbf{r} = \mathbf{b} - \mathbf{Ax}^*$ be the residual. Prove

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|}{\|\mathbf{x}\|} \leq \kappa(\mathbf{A}) \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$$

where $\|\cdot\|$ is any vector norm, and $\kappa(\mathbf{A})$ is the condition number of $\mathbf{A}$ with respect to the induced matrix norm.
(b) For the matrix

$$\mathbf{A} = \begin{bmatrix} 5.4 & 0.6 & 2.2 \\ 0.6 & 6.4 & 0.5 \\ 2.2 & 0.5 & 4.7 \end{bmatrix},$$

compute an upper bound for the condition number $\kappa_2(\mathbf{A})$, using the estimates of the eigenvalues by the Gershgorin Circle Theorem.

6. Prove that every Hermitian, positive definite matrix $\mathbf{A}$ (i.e., $\mathbf{x}^*\mathbf{Ax} > 0$ for all $\mathbf{x} \neq \mathbf{0}$) has a unique Cholesky factorization (i.e, $\mathbf{A} = \mathbf{R}^*\mathbf{R}$ with $r_{jj} > 0$).

7. Compute one step of the QR algorithm (for computing eigenvalues) with the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

(a) Without shift.
(b) With shift $\mu = 1$.

8. Let $\mathbf{A}$ be a real symmetric positive definite matrix and given a linear system of equations $\mathbf{Ax} = \mathbf{b}$. Consider an iterative solution strategy of the form

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{r}_k.$$

where $\mathbf{x}_0$ is arbitrary, $\mathbf{r}_k = \mathbf{b} - \mathbf{Ax}_k$ is the residual and α_k is a scalar parameter to be determined.
(a) Derive an expression for α_k such that $\|\mathbf{r}_{k+1}\|_2$ is minimized as a function of α_k. Is this expression always well-defined and nonzero?
(b) Show that with this choice

$$\frac{\|\mathbf{r}_k\|_2}{\|\mathbf{r}_0\|_2} \leq \left(1 - \frac{\lambda_{min}(\mathbf{A})}{\lambda_{max}(\mathbf{A})}\right)^{k/2}$$

where $\lambda_{min}(\mathbf{A})$ and $\lambda_{max}(\mathbf{A})$ denote the minimal and maximal eigenvalues of $\mathbf{A}$ respectively.