Qualifying Exam in Computational Math, August 2014

Name:

- No books, notes, calculators, cell phones, PDAs, laptops, or any other aids allowed.
- You must show all work as much as possible. Answers without work, even correct, will receive no credit.

Part I: Numerical Analysis 708

1. (10 pts.) Find the Lagrange and Newton forms of the interpolating polynomials for the following set of data. Write both polynomials in the form $a + bx + cx^2$ in order to verify that they are identical.

x	-1	0	2
$f(x)$	0	1	-1

2. (10 pts.) (a) Construct the Lagrange interpolating polynomial of degree 1, denoted by p_1, for the function $f(x) = (x-a)^3$ using the interpolation points $x_0 = 0$ and $x_1 = a$.

   (b) Show that in the interpolation error $f(x) - p_1(x)$, ξ has the unique value $\xi = (x+a)/3$.

3. (10 pts.) The Newton-Cotes formula with $n = 3$ on the interval $[-1, 1]$ is

$$\int_{-1}^{1} f(x)dx \approx w_0 f(-1) + w_1 f(-1/3) + w_2 f(1/3) + w_3 f(1).$$

   Using the fact that this formula is to be exact for all polynomials of degree 3, find the values of the weights w_0, w_1, w_2, and w_3.

4. (10 pts.) (a) Using Richardson's extrapolation to derive a numerical differentiation formula of order $O(h^4)$ to

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(x) - \frac{h^4}{120} f^{(5)}(x) - \dots$$

   In addition to the formula, provide the leading term in the error (i.e., the coefficient of $O(h^4)$).

   (b) What is the order of accuracy of the following approximation?

$$f'''(x) \approx \frac{1}{2h^3}[f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)]$$

   Using Taylor series expansions to find the error term for this approximation.

5. (10 pts.) To solve ODE $y' = f(t, y)$ with initial value $y(0) = y_0$ we have several numerical methods. Show the formulas of the following methods and their corresponding properties.

   (a) **Euler method**, derive the local truncation error T_n using Taylor expansion. For ODE $y' = \lambda y$ with step size h, what is the domain of absolute stability?

   (b) **Implicit Euler method**, derive the local truncation error T_n using Taylor expansion. For ODE $y' = \lambda y$ with step size h, what is the domain of absolute stability?

   (c) **Trapezoidal method**, derive the local truncation error T_n using Taylor expansion.

   (d) **Improved Euler (Runge-Kutta 2) method**, derive the local truncation error T_n using Taylor expansion.

Part II: Numerical Linear Algebra 709

6. (10 pts.) For $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$

   (a) Find $\|A\|_2$, $\|A\|_1$, $\|A\|_\infty$, and $\|A\|_F$.

   (b) Find SVD factorization of A.

7. (10 pts.) Suppose A is normal, i.e. $AA^* = A^*A$, show that if A is also triangular, it must be diagonal. Use this to show that an $n \times n$ matrix is normal if and only if it has n orthonormal eigenvectors.

8. (10 pts.) Using the Gram-Schmidt iteration, find the QR factorization for:

   $$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

9. (10 pts.) (a) Prove that every Hermitian positive definite matrix A has a unique Cholesky factorization (i.e., $A = R^*R$ with $r_{jj} > 0$).

   (b) Find the Cholesky factorization for:

   $$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$$

10. (10 pts.) Compute one step of the QR algorithm (for computing eigenvalues) with the matrix

   $$A = \begin{bmatrix} 2 & \epsilon \\ \epsilon & 1 \end{bmatrix}$$

   (a) Without shift.

   (b) With shift $\mu = 1$.